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Filon's construct for dislocations and related topics

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Abstract. A unified account is presented of the relationships in plane homogeneous isotropic linear elasticity between dislocations, thermoelasticity, inclusions, and the variation of Poisson's ratio. While basic principles are emphasised, there is also indicated how solutions in one theory may be used to generate solutions in another. The connexion between dislocations and the variation of Poisson's ratio, and between dislocations and thermoelasticity are due respectively to Filon and Mushkelishvili.

Key words: dislocations, inclusions, plane linear elasticity, thermoelasticity, variation of Poisson's ratio

1. Introduction

Filon [1], in his paper on plane isotropic linear elasticity to the British Association for the Advancement of Science held at Edinburgh in 1921, described how the difference in two solutions to the same traction boundary value problem with the same shear modulus but different Poisson ratios is related to the solution to the problem of an edge dislocation with zero boundary traction. He applied the result principally to photoelasticity (see Coker and Filon [3, p. 518]). Muskhelishvili [2, p. 166] later analysed a second connexion, again in plane elasticity, between a dislocation solution and that for a corresponding problem in plane thermoelasticity.

A commemorative volume dedicated to Filon furnishes the opportunity for a unified evaluation of these and related contributions. For convenience, only the homogeneous theory is considered. A complex-variable formulation, focussed on the variation of Poisson's ratio, facilitates a systematic investigation and clarifies the extension to other problems. The main intention is not to present new results, although some are included. Instead, the account seeks to extract and recall basic principles useful not only for theoretical development but also for the generation of solutions to specific boundary-value problems in the plane theories of dislocations, thermoelasticity and elastic inclusions. In this respect, the relationships due to Filon and Muskhelishvili, apparently frequently overlooked, help to avoid unnecessary duplication.

Several topics discussed here are alternatively treated in the book by Timoshenko and Goodier [4].

Section 2 assembles well-known elements of plane strain elasticity needed subsequently. In particular, the necessary and sufficient conditions first established by Michell [5] for the stress to be independent of the elastic moduli are recalled. Section 3 employs complex variables to express the difference in the displacement for two different Poisson ratios but for the same boundary conditions and shear modulus from which the relationships of Filon and Muskhelishvili immediately follow. Simple examples illustrate the interrelationships and the procedure for generating exact solutions in the respective theories. Section 4 examines the inclusion problem of a homogeneous matrix containing one or more bonded inclusions of different Poisson ratios. The solution is derived from that to the problem in which Poisson's ratio alters its value in those parts of an initially homogeneous material subsequently occupied by the inclusions. The relationships, both for a heated inclusion in a cold homogeneous matrix and for a distribution of Somigliana dislocations, are easily derived. The second relationship also links with the approach to inclusions adopted by Eshelby [6] and others. A surprising feature is that the solution reduces solely to a dependence upon that for the homogeneous problem with the inclusions absent. In conclusion, Section 5 consists of some brief remarks that include extension of the procedure to allow a variation in the shear modulus as well as in Poisson's ratio.

A suitably smooth solution is always assumed to exist for the problems treated. The conventions of summing over repeated suffixes, and of a subscript comma to denote partial differentiation are adopted throughout. Greek lower case letters range over the values 1,2.

2. Elements of basic theory

2.1. General

Presented without proof in this section are selected well-known results from the complex variable theory of isotropic homogeneous plane-strain elasticity that are required subsequently. Standard references include the books by Green and Zerna [7], Milne-Thomson [8], Muskhelishvili [2], Sokolnikoff [9], and Timoshenko and Goodier [4].

Consider an infinitely long prismatic cylinder whose connected cross-section may be bounded or unbounded , and either singly or multiply connected. Select a cross-section Ω, and let x_1, x_2 be the coordinates of a point in Ω with respect to a two-dimensional Cartesian system whose origin is located in the plane of Ω. Introduce the complex variable $z = x_1 + \mathrm{i}x_2$ together with its complex conjugate $\overline{z} = x_1 - \mathrm{i}x_2$, and define complex differentiation by:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right) \quad , \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right). \tag{2.1.1}$$

The cylinder is occupied by a homogeneous isotropic linear elastic material of Lamé constants λ and μ related to the Poisson ratio ν for $2\nu \neq 1$ by

$$\lambda = \frac{2\mu\nu}{1-2\nu}\,. \tag{2.1.2}$$

Equilibrium is maintained under zero body force and prescribed boundary conditions that produce a plane-strain deformation in the cross-section Ω. In those parts of Ω where the Cartesian components (u_1, u_2) of the displacement are differentiable, the strain and rotation are defined respectively by the relations:

$$e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \ , \qquad \alpha, \beta = 1, 2, \tag{2.1.3}$$

$$\omega = \frac{1}{2}\left(u_{2,1} - u_{1,2}\right) \ . \tag{2.1.4}$$

The strain, where defined, is supposed single valued, but the displacement is assumed single valued only when the region Ω is singly connected. Conditions under which the displacement is single valued in multiply connected regions are discussed later.

Let $D = u_1 + \mathrm{i}u_2$. Then

$$\frac{\partial D}{\partial z} = \frac{1}{2} e_{\alpha\alpha} + \mathrm{i}\omega, \tag{2.1.5}$$

$$\frac{\partial \overline{D}}{\partial z} = \frac{1}{2} (e_{11} - e_{22}) - \mathrm{i}e_{12}, \tag{2.1.6}$$

or by rearrangement,

$$e_{\alpha\alpha} = \frac{\partial D}{\partial z} + \frac{\partial \overline{D}}{\partial \overline{z}}, \qquad \mathrm{i}\omega = \frac{1}{2}\left(\frac{\partial D}{\partial z} - \frac{\partial \overline{D}}{\partial \overline{z}}\right), \tag{2.1.7}$$

$$2\mathrm{i}e_{12} = \left(\frac{\partial D}{\partial \overline{z}} - \frac{\partial \overline{D}}{\partial z}\right), \tag{2.1.8}$$

$$e_{11} = \frac{1}{2}\left(\frac{\partial D}{\partial z} + \frac{\partial \overline{D}}{\partial \overline{z}}\right) + \frac{1}{2}\left(\frac{\partial \overline{D}}{\partial z} + \frac{\partial D}{\partial \overline{z}}\right), \qquad e_{22} = \frac{1}{2}\left(\frac{\partial D}{\partial z} + \frac{\partial \overline{D}}{\partial \overline{z}}\right) - \frac{1}{2}\left(\frac{\partial \overline{D}}{\partial z} + \frac{\partial D}{\partial \overline{z}}\right), \tag{2.1.9}$$

where appeal has been made to the relation

$$\frac{\partial \overline{D}}{\partial \overline{z}} = \overline{\frac{\partial D}{\partial z}}. \tag{2.1.10}$$

The stress-tensor components $\sigma_{\alpha\beta}$ with respect to the chosen Cartesian coordinate axes are assumed differentiable and single valued irrespective of whether the region Ω is simply or multiply connected, and are expressed in terms of the strain components by the constitutive relations

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}, \tag{2.1.11}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Substitution of (2.1.11) in the stress equilibrium equations

$$\sigma_{\alpha\beta,\beta} = 0 \tag{2.1.12}$$

enables the introduction of complex analytic potential functions $\varphi(z)$ and $\psi(z)$ that are holomorphic (*i.e.*, analytic and single valued) in singly connected regions Ω. In terms of the complex potentials, the displacement and stress components are expressed by

$$2\mu D = (3 - 4\nu)\,\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{2.1.13}$$

$$\sigma_{\alpha\alpha} = 2\left(\varphi'(z) + \overline{\varphi'(z)},\right), \tag{2.1.14}$$

$$\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2\left[\overline{z}\varphi''(z) + \psi'(z)\right], \tag{2.1.15}$$

where a superposed prime indicates differentiation with respect to the argument of the relevant function. In addition the notation

$$\overline{F(z)} = \overline{F}(\overline{z}), \tag{2.1.16}$$

is adopted, so that in particular

$$\overline{\varphi'(z)} = \frac{\partial \overline{\varphi}(\overline{z})}{\partial \overline{z}}. \tag{2.1.17}$$

From (2.1.7) and (2.1.13) follows the identity

$$(\lambda + 2\mu)\, e_{\alpha\alpha} + 2\mathrm{i}\mu\omega = 2\frac{(\lambda + 2\mu)}{(\lambda + \mu)}\varphi'(z)\,, \tag{2.1.18}$$

which implies that $(\lambda + 2\mu)\, e_{\alpha\alpha}$ and $2\mathrm{i}\mu\omega$ are the real and imaginary parts of an analytic complex function and accordingly $e_{\alpha\alpha}$ and ω vanish together. It is supposed that the displacement and traction are continuous onto any smooth non-intersecting curve C in the region Ω and also onto the boundary $\partial\Omega$. Further, the solution is assumed regular in the sense of Muskhelishvili [2, p. 155], namely, that the complex potentials $\varphi(z)$ and $\psi(z)$ and the derivative $\varphi'(z)$ are assumed to be continued continuously at all points of the external (and internal) boundary $\partial\Omega$. The traction with Cartesian components (F_1, F_2) across the curve C at any point $z(s)$, where s is the arc-length along C, possesses the equivalent representations:

$$F_1 + \mathrm{i}F_2 = -\mathrm{i}\left[(\lambda + 2\mu)\, e_{\alpha\alpha} + 2\mathrm{i}\mu\omega\right]\frac{\mathrm{d}z}{\mathrm{d}s} + 2i\mu\frac{\mathrm{d}D}{\mathrm{d}s} \tag{2.1.19}$$

$$= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\left[4(1-\nu)\,\varphi(z) - 2\mu D\right] \tag{2.1.20}$$

$$= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right]. \tag{2.1.21}$$

When C is a simple closed curve, it follows from (2.1.21) that the total force acting across C is given by

$$\oint_C (F_1 + \mathrm{i}F_2)\,\mathrm{d}s = -\mathrm{i}\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right]_C, \tag{2.1.23}$$

where $[\cdot]_C$ denotes the increase in the enclosed quantity as the curve C is described once in the positive (anti-clockwise) direction. For simplicity, attention is restricted to either displacement or traction boundary conditions on $\partial\Omega$. Consequently, either

$$D = f, \qquad z \in \partial\Omega, \tag{2.1.23}$$

or

$$F_1 + \mathrm{i}F_2 = g, \qquad z \in \partial\Omega, \tag{2.1.24}$$

are assumed, where f and g are given complex functions on $\partial\Omega$. When Ω is unbounded, the asymptotic behaviour as $|z| \to \infty$ of either the displacement or stress and rotation must be specified. In terms of the elastic moduli, necessary and sufficient conditions for the uniqueness of the stress and displacement in the displacement boundary-value problem are that

$$-\infty \le \nu \le \frac{1}{2}, \qquad 1 < \nu \le \infty, \qquad \mu \ne 0, \tag{2.1.25}$$

while in the traction boundary problem, the stress is unique if and only if $\nu \ne 1$ while the displacement is unique if and only if $\nu \ne 1$, $\mu \ne 0$. Both sets of conditions may be relaxed for special geometries. For example, when Ω is a circle, a necessary and sufficient condition in the displacement boundary-value problem is $\nu \neq 1, 3/4$, $\mu \neq 0$. Details of these and other results may be found in [10, Chapter 5, p. 61].

2.2. Singly connected regions

Selected properties of the solution are recalled when the region Ω is singly connected. Multiply connected regions are considered in the next subsection. First, observe that since the displacement is single valued in a singly connected region (2.1.13) leads to

$$\left[(3-4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right]_C = 0, \tag{2.2.1}$$

for any simple closed curve C in the region Ω. The total traction on C is zero for equilibrium and (2.1.22) gives

$$\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right]_C = 0. \tag{2.2.2}$$

Expressions (2.2.1) and (2.2.2) imply that $[\varphi(z)]_C = 0$, unless $\nu = 1$, and consequently $\varphi(z)$ is holomorphic in Ω. But differentiation inside the square bracket yields $\left[\varphi'(z)\right]_C = 0$, which together with (2.2.2) shows that $\left[\psi(z)\right]_C = 0$ and $\psi(z)$ is likewise holomorphic in Ω.

The complex potentials are determined from the boundary conditions (2.1.23) or (2.1.24) either by means of power-series expansions or by the theory of Cauchy integrals. Details are presented in standard texts and demonstrate that, in principle at least, any boundary-value problem may be solved explicitly. A particular property of these solutions, of subsequent significance, is that for the traction boundary-value problem the complex potentials, determined from (2.1.21) and (2.1.24), are independent of the elastic moduli. Moreover, the stress components, being given by (2.1.14) and (2.1.15) similarly are insensitive to variations in the Poisson ratio and shear modulus (*cp.*, [2, p. 160]). The displacement, given by (2.1.13), obviously does not exhibit the same property.

2.3. Multiply connected regions. Dislocations

Consider a multiply connected region Ω and allow the displacement to be multi-valued while still requiring the stress components to remain single valued. Let Ω be bounded internally by n smooth non-intersecting closed contours $\partial\Omega_k$, $k = 1, \ldots, n$, and externally by the boundary $\partial\Omega_0$ (when the region is bounded). According to Milne-Thomson [8], the jump discontinuity in the displacement around any simple closed curve taken in the positive (anti-clockwise) direction may be calculated as follows.

By virtue of the single valuedness of the stress components, the relations (2.1.14) and (2.1.15) imply the expressions

$$\left[\varphi'(z)\right]_C + [\overline{\varphi'(z)}]_C = 0, \tag{2.3.1}$$

$$\left[\bar{z}\varphi''(z)\right]_C + \left[\psi'(z)\right]_C = 0, \tag{2.3.2}$$

around any smooth closed contour C in Ω. Differentiation and integration commute with the operation of jump discontinuity, so that (2.3.1) yields $\left[\varphi''(z)\right]_C = 0$, and consequently

$$[\varphi(z)]_C = \mathrm{i}zA + B, \tag{2.3.3}$$

in which A and B are a real and complex constant, respectively. Similarly, (2.3.2) implies

$$\left[\psi(z)\right]_C = E, \tag{2.3.4}$$

for complex constant E. Insertion of (2.3.3) and (2.3.4) into (2.1.13) gives the required expression for the jump discontinuity of the displacement around the closed curve C:

$$2\mu\,[D]_C = 4(1-\nu)\mathrm{i}zA + (3-4\nu)B - E. \tag{2.3.5}$$

Such discontinuities are known as (Volterra) dislocations and consist of an edge dislocation, corresponding to the rigid body translation $[(1-4\nu)B - E]$, and what is termed a wedge disclination, corresponding to the rigid-body rotation through the (small) angle $4(1-\nu)A$. Dislocations in the present context are discussed in further detail in, for example, the books [8], [2, p. 162], [11, p. 221], and [12]. Filon [1] (see also Coker and Filon[3, p. 518]) describes the relation between edge dislocations and the difference of two single-valued displacements in problems having the same boundary traction and shear modulus but different Poisson ratios. Section 3 rederives the conclusion as part of an integrated account.

This sub-section, meanwhile, is concluded with an examination of conditions for boundary-value problems in the multiply-connected region Ω to possess a single-valued displacement and stress components that are independent of the elastic moduli. For a simply connected region, it has been noted in Section 2.2 that such independence can occur only in the traction boundary problem. In a multiply connected region, the same argument shows that the stress in the traction boundary-value problem is likewise independent of the elastic moduli but allows the displacement to be multi-valued. Denote the resultant traction across each internal boundary $\partial\Omega_k$ described *clockwise* by $X_k + \mathrm{i}Y_k$ and let A_k, B_k, and E_k be the respective constants appearing in (2.3.3) and (2.3.4). Relation (2.1.22) leads to:

$$X_k + \mathrm{i}Y_k = \mathrm{i}(B_k + \mathrm{i}E_k). \tag{2.3.6}$$

Now restrict the displacement to be single valued. From (2.3.5) for each k it may be concluded that:

$$A_k = 0, \qquad (3-4\nu)B_k - E_k = 0, \tag{2.3.7}$$

which together with (2.3.6) implies that:

$$B_k = \frac{(Y_k - \mathrm{i}X_k)}{4(1-\nu)}, \tag{2.3.8}$$

$$E_k = \frac{(3-4\nu)(Y_k + \mathrm{i}X_k)}{4(1-\nu)}. \tag{2.3.9}$$

Let $(\nu^{(1)}, \mu^{(1)})$ and $(\nu^{(2)}, \mu^{(2)})$ be different moduli and let φ and ψ be complex potentials for the traction boundary-value problem with moduli $(\nu^{(1)}, \mu^{(1)})$ and consequently for the same traction boundary-value problem with moduli $(\nu^{(2)}, \mu^{(2)})$. For each contour $\partial\Omega_k$, the constants A_k, B_k, E_k likewise remain unaltered and satisfy the relations (2.3.7)–(2.3.9) for both sets of moduli. But this is possible only when $X_k = Y_k = 0$,for $k = 1, \ldots, n$. Consequently, in the traction boundary-value problem for a multiply connected region, the displacement can be single valued and the stress components independent of the elastic moduli if and only if the resultant traction is zero over each internal boundary $\partial\Omega_k$ (and by overall equilibrium also over $\partial\Omega_o$). The conclusion is invalidated when multi-valued displacements are admitted.

The result was first derived by Michell [5] (see also [2, p. 161]). Corresponding investigations in three dimensions by Carlson [13] (see also Dundurs [14] and Sternberg and Muki [15]) establish, for example, that the stress is independent of Poisson's ratio if and only if the dilatation vanishes for at least one value of Poisson's ratio. Other results are obtained with respect to the shear modulus, while estimates for the continuous dependence of the solution on the elastic moduli have been constructed by Bramble and Payne [16]; see also Knops and Payne [17]. These studies are related to the notion in nonlinear elasticity of a universal solution valid irrespective of material properties.

2.4. THERMOELASTICITY

Let the cylinder be occupied by an isotropic homogeneous linear thermoelastic material subject to a uniform steady temperature distribution and let the cross section Ω experience a plane strain deformation. In the absence of heat sources, the temperature $T(x_1, x_2)$, supposed independent of the cylinder's axial variable and single valued, is an harmonic function. The relevant constitutive relations are:

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} - (3\lambda + 2\mu)\kappa T\delta_{\alpha\beta}, \tag{2.4.1}$$

where κ is the constant coefficient of linear thermal expansion. Equilibrium is maintained under zero body force, displacement and traction boundary conditions (2.1.23) or (2.1.24), and thermal boundary conditions consisting of either

$$T(x_1, x_2) = T_1, \qquad (x_1, x_2) \in \partial\Omega, \tag{2.4.2}$$

or

$$n_\alpha T,_\alpha + h(T - T_0) = T_2, \qquad (x_1, x_2) \in \partial\Omega, \tag{2.4.3}$$

in which (n_1, n_2) are the cartesian components of the unit outward normal on $\partial\Omega$, T_1 and T_2 are specified functions of x_1 and x_2 on $\partial\Omega$, T_0 is the ambient temperature of material surrounding Ω, and h is the known coefficient of heat transfer on $\partial\Omega$ between Ω and its exterior.

It is proved in the standard texts (see also Lurie [18]) that the thermal displacement is given by

$$2\mu D = (3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \mu\kappa\Theta\frac{(3\lambda + 2\mu)}{(\lambda + \mu)}, \tag{2.4.4}$$

where

$$\Theta = \int (T + \mathrm{i}S)\, \mathrm{d}z, \tag{2.4.5}$$

and $S(x_1, x_2)$ is the imaginary part of the complex function whose real part is $T(x_1, x_2)$. (The function S may be multi-valued even when T is single valued.) The complex potentials $\varphi(z)$ and $\psi(z)$ retain the previously described properties and, in fact, the stress components continue to be given by (2.1.14.) and (2.1.15). It is immediate in the traction boundary-value problem that the stress is independent of the temperature and elastic moduli (*cp.*, [2, pp 165–167], [9, p. 366]).

The assemblage of elements from the basic theory is now complete. The next section discusses how plane theories of edge dislocations, thermoelasticity, and variation of Poisson's ratio may be interrelated. Section 4 considers the problem of a matrix containing embedded bonded inclusions of different Poisson ratios and also discusses the relation with thermoelasticity.

3. Interrelationships

3.1. Introduction

Solutions in isothermal plane elasticity are considered for which boundary conditions and the shear modulus remain unaltered but Poisson's ratio varies from one uniform value to another. First, it is explained how the difference between two such solutions may be employed to solve new problems, and then the main task is addressed of relating the difference solution to ones in the plane theory of edge dislocations and thermoelasticity. This enables solutions in one theory to be derived from those in another or from an isothermal problem for which Poisson's ratio has been selected to simplify the calculations. The respective procedures are illustrated by examples that mainly concern singularities.

3.2. Notation

Let quantities corresponding to isothermal solutions with Poisson ratios $\nu^{(\alpha)}, \alpha = 1, 2$, be denoted by superscripts so that, for example, the respective complex potentials are $\varphi^{\alpha)}(z)$ and $\psi^{(\alpha)}(z)$. Define the differences between the complex potentials to be $\varphi = \varphi^{(1)} - \varphi^{(2)}$, $\psi = \psi^{(1)} - \psi^{(2)}$; and that between the displacement and stress components to be:

$$D = D^{(1)} - D^{(2)}, \tag{3.2.1}$$

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{(1)} - \sigma_{\alpha\beta}^{(2)}. \tag{3.2.2}$$

From (2.1.13)–(2.1.15) it follows that

$$2\mu D = (3 - 4\nu^{(1)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - 4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z), \tag{3.2.3}$$

$$\sigma_{\alpha\alpha} = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{3.2.4}$$

$$\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2[\overline{z}\varphi''(z) + \psi'(z)], \tag{3.2.5}$$

where μ is the unaltered shear modulus. The difference boundary conditions reduce to either

$$D = 0, \qquad z \in \partial\Omega, \tag{3.2.6}$$

or

$$F_1 + \mathrm{i}F_2 = 0, \qquad z \in \partial\Omega, \tag{3.2.7}$$

depending upon whether the boundary displacement or boundary traction is specified initially.

The difference complex potentials φ and ψ inherit properties enjoyed by $\varphi^{(\alpha)}$ and $\psi^{(\alpha)}$. In particular, φ and ψ vanish identically in the traction boundary problem under conditions stated in Sections 2.2 and 2.3. Nevertheless, the difference displacement D does not vanish in similar circumstances and indeed may contain dislocations when Ω is multiply connected. Before embarking on further discussion of these and related topics, we investigate briefly how the formulation (3.2.3)–(3.2.7) may be used to solve isothermal problems.

3.3. Isothermal problems

In order to employ the difference displacement and stress to derive a solution to an isothermal boundary-value problem for a general Poisson ratio, it is necessary to first select a value of the Poisson ratio that simplifies the calculation and enables the complex potentials $\varphi^{(2)}$ and $\omega^{(2)}$ to be determined easily. Certain simplifying values are discussed in [10, Chapter 3, p. 23]. The selected Poisson ratio need not yield a unique solution; for example, the values $\nu^{(2)} = 1$ or $\frac{3}{4}$ can be included. The customary modification, however, is required for an incompressible material when $\nu^{(2)} = \frac{1}{2}$. For such problems, it worth recalling the connexion established by Hill [19] that relates solutions to the displacement and traction boundary problems. In singly connected regions, the stress is independent of the elastic moduli for a given traction boundary problem and consequently by Hill's analogy may be found from the solution to the corresponding displacement boundary-value problem in the incompressible case. Further, in this category of problems, the difference complex potentials are identically zero or constant, and the difference displacement, which may be regarded as occurring in a material of Poisson ratio $\nu^{(1)}$, is given by

$$2\mu D = -4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z), \qquad z \in \Omega. \tag{3.3.1}$$

On the other hand, for the displacement boundary problem, insertion into the boundary condition (3.2.6) of the known complex potential $\varphi^{(2)}(z)$ from the simplified problem with Poisson ratio $\nu^{(2)}$ leads to:

$$(3 - 4\nu^{(1)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = 4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z), \qquad z \in \partial\Omega, \tag{3.3.2}$$

from which the difference complex potentials φ and ψ may be determined by standard methods provided the general value $\nu^{(1)}$ of Poisson's ratio lies in the uniqueness range. The final complex potentials $\varphi^{(1)}$ and $\psi^{(1)}$ for Poisson ratio $\nu^{(1)}$ are obtained from the relations $\varphi^{(1)} = \varphi^{(2)} + \varphi, \quad \psi^{(1)} = \psi^{(2)} + \psi$.

The procedure, however, offers little obvious advantage over standard methods of solution in plane elasticity. Nevertheless, it can be effective in three-dimensions where it has successfully solved new problems (*e.g.*, Knops [20]) and represents a generalisation of Westergaard's twin-gradient method [21].

3.4. Dislocations

For a simply connected region Ω the complex potentials $\varphi^{(\alpha)}$ and $\psi^{(\alpha)}$ are holomorphic and consequently so also are the difference complex potentials φ and ψ. This conclusion implies that the difference displacement D cannot contain a dislocation and that an interpretation of the difference displacement in terms of dislocations can be sought only for regions Ω that are multiply connected. Accordingly, let Ω be bounded internally by n smooth non-intersecting closed contours $\partial\Omega_k, k = 1, 2, \dots, n$, over each of which the resultant traction $X_k + iY_k$ is non-zero, and further suppose that the displacement $D^{(\alpha)}$ and stress $\sigma^{(\alpha)}_{\alpha\beta}$ are single valued in Ω. Note that Michell's result (see Section 2.3) implies that the stress necessarily depends upon the elastic moduli. By virtue of the single valuedness of the displacement $D^{(2)}$, the relations (2.3.3) and (2.3.8) lead to the expression

$$\left[\varphi^{(2)}(z)\right]_{\partial\Omega_k} = \frac{\mathrm{i}(X_k + \mathrm{i}Y_k)}{4(1-\nu^{(2)})}, \tag{3.4.1}$$

where $\partial\Omega_k$ is described clockwise.

Let

$$2\mu D^* = -4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z), \qquad z \in \Omega, \tag{3.4.2}$$

and define $D^\dagger = D - D^*$. Now, D is single valued in Ω and consequently for each k there holds:

$$\mu\left[D^\dagger\right]_{\partial\Omega_k} = -\mu\left[D^*\right]_{\partial\Omega_k} \tag{3.4.3}$$

$$= 2(\nu^{(1)} - \nu^{(2)})[\varphi^{(2)}(z)]_{\partial\Omega_k} \tag{3.4.4}$$

$$= \frac{\mathrm{i}(\nu^{(1)} - \nu^{(2)})(X_k + \mathrm{i}Y_k)}{2(1-\nu^{(2)})}, \tag{3.4.5}$$

after an appeal to (3.4.2) and (3.4.1). It therefore follows that the difference stress distribution given by (3.2.4) and (3.2.5) are associated with a displacement $D^\dagger$ that contains an edge dislocation of amount (3.4.5).

The result was first obtained by Filon [1] (see also Coker and Filon [3, p. 518]) and was used by him to generate solutions in a multiply connected region to the traction boundary problem with a single valued displacement from that containing a specified edge dislocation under zero boundary tractions. The problem is first solved for a simplifying Poisson ratio $\nu^{(1)}$ subject to non-zero tractions $X_k + \mathrm{i}Y_k$ on each boundary $\partial\Omega_k$. Photoelastic experiment is next used to measure the displacement $D^\dagger$ and stress components $\sigma_{\alpha\beta}$ in the same body but with each boundary $\partial\Omega_k$ and $\partial\Omega_o$ traction-free, and with dislocations of amount (3.4.5) introduced between each respective contour $\partial\Omega_k$ and $\partial\Omega_o$. The stress components for any other value of Poisson's ratio $\nu^{(2)}$ in the loaded problem are finally obtained from $\sigma^{(2)}_{\alpha\beta} = \sigma^{(1)}_{\alpha\beta} - \sigma_{\alpha\beta}$, while the associated displacement is given by $D^{(2)} = D^{(1)} - D^\dagger - D^*$. Apart from photoelasticity, the solutions to several other types of dislocation problems are discussed, for example, by Milne-Thomson [8] and Nabarro [12].

The procedure may be reversed to generate solutions to problems containing edge dislocations and illustrative examples are presented in Section 3.7 after the relationship with thermoelasticity is established in the next subsection. Note, however, that the procedure can deal only with edge dislocations and other types of dislocations are excluded.

3.5. THERMOELASTICITY

A second interpretation of the difference displacement and stress distribution is realised on comparing expressions (3.2.3) and (2.4.4) for the displacement produced by steady heat conduction in the plane strain deformation of an elastic material with Poisson ration $\nu^{(1)}$ and shear modulus μ. The expressions become identical on setting

$$4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z) = -\frac{\mu\kappa\Theta(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + \mu)}, \tag{3.5.1}$$

and it may be concluded that the single valued temperature $T(x_1, x_2)$ is related to the isothermal complex potential $\varphi^{(2)}(z)$ by

$$T + \mathrm{i}S = -\frac{4(\nu^{(1)} - \nu^{(2)})(\lambda^{(1)} + \mu)\varphi^{(2)'}(z)}{\mu\kappa(3\lambda^{(1)} + 2\mu)} \tag{3.5.2}$$

$$= -\frac{2(\nu^{(1)} - \nu^{(2)})(\lambda^{(1)} + \mu)(\lambda^{(2)} + \mu)[(\lambda^{(2)} + 2\mu)e^{(2)}_{\gamma\gamma} + 2\mathrm{i}\mu\omega^{(2)}]}{\mu\kappa(3\lambda^{(1)} + 2\mu)(\lambda^{(2)} + 2\mu)}, \tag{3.5.3}$$

by virtue of (2.1.18)

A relationship therefore exists between a plane thermoelastic problem with steady heat conduction subject to zero mechanical boundary conditions and the difference of the solutions to two isothermal plane elastic problems for the same boundary conditions and shear modulus but different Poisson ratios. When the solution for arbitrary Poisson ratio is known for the isothermal problem, the temperature is given by (3.5.2) for thermoelastic material of elastic moduli $\nu^{(1)}$, μ and coefficient of thermal expansion κ. The mechanical boundary conditions in the thermal problem consist of either zero displacement or zero traction depending upon whether the boundary displacement or boundary traction are specified in the isothermal problem. An analogous procedure in three dimensions has been developed by Knops [22].

The thermoelastic complex potentials are φ and ψ and lead to expressions (3.2.3)–(3.2.5) for the thermoelastic displacement and stress. The thermal boundary conditions generated by this procedure are obtained by inserting the expression for T given by (3.5.2) into either the boundary conditions (2.4.2) or (2.4.3) and deriving either T_1, h or T_2 in semi-inverse fashion. In general, the thermal boundary conditions cannot be prescribed *a priori*. Conversely, a known plane thermoelastic solution may be used to generate an isothermal one by first selecting an appropriate Poisson ratio $\nu^{(2)}$ and calculating the corresponding complex potential $\varphi^{(2)}(z)$ from (3.5.1) or (3.5.2). The second complex potential $\psi^{(2)}(z)$ is determined from the prescribed mechanical boundary conditions and the general isothermal solution obtained by adding the thermoelastic solution to that calculated from the complex potentials $\varphi^{(2)}(z)$ and $\psi^{(2)}(z)$. When the isothermal dilatation or rotation vanish for a particular value of Poisson's ratio $\nu^{(2)}$ then relation (3.5.3) implies that the temperature must also vanish in the corresponding thermal problem, where by construction the mechanical boundary conditions are zero. Accordingly, for a Poisson ratio $\nu^{(1)}$ that lies in the range for the solution to be unique it follows that the stress and strain in the thermal problem are identically zero. This in turn implies that the stress and strain in the isothermal problem are independent of Poisson's ratio. The conclusion has been established in three dimensions by Carlson [13] by different means.

More generally, the isothermal traction boundary-value problem in a simply connected region produces a stress distribution independent of the elastic moduli (Section 2.2). Consequently, the stress must vanish identically in the corresponding thermoelastic problem having a steady temperature $T(x_1, x_2)$ and zero-traction boundary conditions. Now assume that for any given $T(x_1, x_2)$ suitable isothermal traction boundary conditions can be found for Poisson ratio $\nu^{(2)}$ such that the corresponding complex potential $\varphi^{(2)}(z)$ satisfies (3.5.1). The thermal displacement is given by (3.2.3) with the complex potentials φ and ψ identically zero. In summary, the stress vanishes identically in any plane steady heat conduction problem subject to zero traction boundary conditions and in a simply connected region. Similar remarks apply to multiply connected regions under conditions outlined in Section 2.3. The conclusion appears first to have been noted by Muskhelishvili [2, p. 168].

3.6. Relationship between dislocations and thermolasticity

Admission of dislocations into the discussion necessarily requires the region Ω to be multiply connected. As before, let there be n internal boundaries $\partial\Omega_k, k = 1, \dots, n$, satisfying the previously stated conditions and suppose that Ω is either bounded or unbounded. From Sections 3.4 and 3.5 it follows that

$$2\mu D^* = -4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z) = \frac{\mu\kappa\Theta(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + \mu)}, \tag{3.6.1}$$

which immediately establishes the connexion between dislocations, steady-state plane thermoelastic problems under zero boundary conditions, and the difference between isothermal plane elastic solutions for different Poisson ratios but the same boundary conditions and shear modulus. Observe that in the multiply connected region, although T is assumed single valued, the function Θ defined by (3.5.1) may be multi-valued providing conditions for dislocations to occur. The explicit relationship (3.6.1) between dislocations and thermoelasticity in the plane theory is due to Muskhelishvili [2, p. 165–170] who adopted, however, a slightly different derivation. (See also [23].) Solutions, as already explained, can be converted from one set of problems into either of the other two. The procedure is illustrated in the next section by simple examples.

3.7. Examples

The following examples, from the many available in the literature, are selected to illustrate the application of the procedure to problems in which certain singularities occur.

3.7.1. *Point force in the infinite plane*

Consider a point force $X_1 + \mathrm{i}X_2$ externally applied at the origin in the infinite plane and suppose that the stress uniformly vanishes at infinity. The well-known complex potentials for a general Poisson ratio ν and shear modulus μ are:

$$\varphi(z) = -\frac{(X_1 + \mathrm{i}X_2)\log z}{8\pi(1-\nu)}, \qquad \psi(z) = \frac{(3-4\nu)(X_1 - \mathrm{i}X_2)\log z}{8\pi(1-\nu)}. \tag{3.7.1}$$

We first discuss the dislocation problem, and choose arbitrary values $\nu^{(1)}$ and $\nu^{(2)}$ of Poisson's ratio. For any smooth curve C enclosing the origin and described anti-clockwise, the edge dislocation in the displacement $D^\dagger$ from (3.4.5) is given by:

$$\left[D^\dagger\right]_C = b_1 + \mathrm{i}b_2, \tag{3.7.2}$$

where

$$b_1 = -\frac{(\nu^{(1)} - \nu^{(2)})X_2}{2\mu(1-\nu^{(2)})}, \tag{3.7.3}$$

$$b_2 = \frac{(\nu^{(1)} - \nu^{(2)})X_1}{2\mu(1-\nu^{(2)})}, \tag{3.7.4}$$

while by equilibrium across C there holds:

$$X_1 + \mathrm{i}X_2 = -\oint_C (F_1 + \mathrm{i}F_2)\mathrm{d}s. \tag{3.7.5}$$

The difference complex potentials (in a notation not to be confused with that in (3.7.1)) are given by

$$\varphi(z) = \frac{\mathrm{i}\mu(b_1 + \mathrm{i}b_2)\log z}{4\pi(1-\nu^{(1)})}, \tag{3.7.6}$$

$$\psi(z) = -\frac{\mathrm{i}\mu(b_1 - \mathrm{i}b_2)\log z}{4\pi(1-\nu^{(1)})}, \tag{3.7.7}$$

and consequently the displacement $D^\dagger$ from (3.2.3) and (3.4.2) becomes

$$D^\dagger = \frac{(\mathrm{i}b_1 - b_2)[(3-4\nu^{(1)})\log z - \log \overline{z}] + (\mathrm{i}b_1 + b_2)z(\overline{z})^{-1}}{8\pi(1-\nu^{(1)})}, \tag{3.7.8}$$

which is the expression otherwise obtained, for example, by Nabarro [12, p. 55]. With respect to the thermoelastic problem, the steady temperature corresponding to the complex potential (3.5.1) from (3.5.2) is given by

$$T = -\frac{2(\nu^{(1)} - \nu^{(2)})(\lambda^{(1)} + \mu)(\varphi^{(2)'}(z) + \overline{\varphi}^{(2)'}(\bar{z}))}{\mu\kappa(3\lambda^{(1)} + 2\mu)} = \frac{m_1 x_\alpha X_\alpha}{4\pi\kappa r^2} \tag{3.7.9}$$

$$m_1 = \frac{\nu^{(1)} - \nu^{(2)}}{(1+\nu^{(1)})(1-\nu^{(2)})}, \qquad r^2 = z\overline{z}. \tag{3.7.10}$$

In terms of the edge dislocation (3.7.2), the temperature may equivalently be expressed as

$$T = \frac{x_1 b_2 - x_2 b_1}{2\pi\kappa(1+\nu^{(1)})}. \tag{3.7.11}$$

The temperature distribution (3.7.9) represents a pair of heat dipoles at the origin in the co-ordinate directions and of strengths $m_1 X_1/2\kappa$ and $m_1 X_2/2\kappa$, respectively. Expressions for the corresponding displacement and stress, obtained from (3.2.3)–(3.2.5) for the complex potentials (3.7.6) and (3.7.7), are omitted.

3.7.2. *The half-plane*

Let Ω be the half-plane $x_2 \le 0$ subject to boundary conditions along the boundary $x_2 = 0$. It is known ([2, p. 464]) that when the stress and rotation vanish uniformly at infinity, the solution for general elastic moduli regardless of boundary conditions is completely represented in terms of the single complex potential $\varphi(z)$, holomorphic in Ω, according to the expressions:

$$2\mu D = (3-4\nu)\varphi(z) + \varphi(\overline{z}) - (z - \overline{z})\overline{\varphi'(z)} + \text{ constant}, \qquad z \in \Omega, \tag{3.7.12}$$

$$\sigma_{\alpha\alpha} = 2[\varphi'(z) + \overline{\varphi'(z)}], \qquad z \in \Omega \tag{3.7.13}$$

$$\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2[(\overline{z} - z)\varphi''(z) - \varphi'(z) - \overline{\varphi}'(z)], \qquad z \in \Omega. \tag{3.7.14}$$

In particular, consider the traction boundary-value problem in which the prescribed pointwise boundary load $F_1(x_1) + \mathrm{i}F_2(x_1)$ on $x_2 = 0$ satisfies certain continuity requirements and possesses a bounded total resultant $X_1 + \mathrm{i}X_2$. For this problem the complex potential is given by:

$$\varphi^{(2)}(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}(F_1(t) + \mathrm{i}F_2(t))\log(t - z)\mathrm{d}t, \qquad z \in \Omega, \tag{3.7.15}$$

where t is a real variable of integration. (A comment is included below on the appropriateness of this class of isothermal problems for solutions to problems containing dislocations.) According to (3.7.9), the present procedure leads to the steady-state temperature:

$$T(x_1, x_2) = -\frac{m_2}{\pi}\int_{-\infty}^{\infty}\left[\frac{F_1(t) + \mathrm{i}F_2(t)}{t - z} + \frac{F_1(t) - \mathrm{i}F_2(t)}{t - \overline{z}}\right]\mathrm{d}t, \tag{3.7.16}$$

in which

$$m_2 = \frac{(\nu^{(1)} - \nu^{(2)})(\lambda^{(1)} + \mu)}{\mu\kappa(3\lambda^{(1)} + 2\mu)}. \tag{3.7.17}$$

Thermal boundary conditions on $x_2 = 0$ are obtained from (3.7.16) on letting $x_2 \to 0$, and may lead to specification of either the temperature or radiation condition. Independence from the elastic moduli of the complex potential (3.7.15) implies that the difference complex potential is identically zero and from (3.7.13) and (3.7.14) that the difference stress distribution likewise vanishes identically. Consequently, there are no thermal stresses $\sigma_{\alpha\beta}$ in the corresponding plane thermoelastic problem subject to a temperature of the form (3.7.16). On the other hand, by (3.7.12), the corresponding difference displacement is expressed by:

$$2\mu D = -\frac{2m_2\mu\kappa(3\lambda^{(1)} + 2\mu)}{\pi(\lambda^{(1)} + \mu)}\int_{-\infty}^{\infty}[F_1(t) + \mathrm{i}F_2(t)]\log(t - z)\mathrm{d}t. \tag{3.7.18}$$

Consider two special loadings ([2, p. 386, p. 395, resp.]). The first is a concentrated point load $X_1 + \mathrm{i}X_2$ at the origin for which

$$\varphi^{(2)}(z) = -\frac{1}{2\pi}(X_1 + \mathrm{i}X_2)\log z. \tag{3.7.19}$$

The corresponding temperature:

$$T(x_1, x_2) = \frac{2m_2}{\pi}\left[X_\alpha\frac{\partial}{\partial x_\alpha}\log r\right], \qquad r^2 = z\overline{z}, \tag{3.7.20}$$

represents a heat dipole at the origin on the boundary of the half-plane, the entire boundary remaining traction-free. The second special load is given by

$$F_1(x_1) = 0, \qquad -\infty < x_1 < \infty, \tag{3.7.21}$$

$$F_2(x_1) = -P, \qquad -a \le x_1 \le a, \tag{3.7.22}$$

$$= 0, \qquad -\infty < x_1 < -a,\, a < x_1 < \infty, \tag{3.7.23}$$

where P is constant, and consists of a uniform pressure P applied over the interval $[-a, a]$. The complex potential is

$$\varphi^{(2)}(z) = \frac{P}{2\mathrm{i}\pi}\left[z\log\frac{(z - a)}{(z + a)} - a\log(z^2 - a^2)\right], \tag{3.7.24}$$

while the corresponding temperature is

$$T(x_1, x_2) = \frac{2m_2 P}{\pi}(\theta_1 - \theta_2), \tag{3.7.25}$$

where $z - a = r_1 \exp(-\mathrm{i}\theta_1)$ and $z + a = r_2 \exp(-\mathrm{i}\theta_2)$. On the boundary, the temperature is given by

$$T(x_1, 0) = 2m_2 P, \qquad -a \le x_1 \le a, \tag{3.7.26}$$

$$= 0, \qquad -\infty < x_1 < -a, a < x_1 < \infty. \tag{3.7.27}$$

The plane thermoelastic problem consists of an interval $[-a, a]$ of the boundary heated to a uniform temperature with the remainder of the boundary held at zero temperature while the entire boundary is traction free. For the previously stated reasons, the thermal stress components $\sigma_{\alpha\beta}$ vanish.

3.7.3. *Half-plane with boundary heat source*

The appropriate isothermal problem, apparently first discussed by Filon [24] (see also Love [11, p. 214]), consists of the half-plane $x_2 \le 0$ loaded on $x_2 = 0$ by a constant tangential force $F_1 = Q$ applied to the semi-infinite interval $-\infty < x_1 \le 0$, with zero traction on the remaining boundary. The complex potential is

$$\varphi^{(2)}(z) = \frac{Q}{2\pi}(z \log z - z), \tag{3.7.28}$$

which produces the temperature distribution

$$T(x_1, x_2) = -\frac{2m_2 Q}{\pi} \log r, \qquad r^2 = z\bar{z}, \tag{3.7.29}$$

where m_2 is given by (3.7.17). Consequently, the plane thermoelastic problem consists of a point heat source located at the origin in an otherwise cold (zero temperature) boundary which is everywhere traction-free. The thermal stress components vanish, while the thermal displacement is

$$2\mu D = -\frac{2Q\mu\kappa m_2(3\lambda^{(1)} + 2\mu)}{\pi(\lambda^{(1)} + \mu)}(z \log z - z). \tag{3.7.30}$$

3.8. FURTHER NOTE ON EDGE DISLOCATIONS AND WEDGE DISCLINATIONS

For the examples in the half-plane considered in the last subsection, it is difficult to envisage circumstances in which the procedure might lead to problems with dislocations. The half-plane is not multiply connected, and the complex potential φ(z) accordingly is holomorphic. It is not obvious what displacement or traction boundary-value problem with uniform temperature and single valued displacement, which upon taking differences for different Poisson ratios, might lead to a solution with dislocations, and in particular, for a boundary distribution of edge dislocations discussed, for example, by Maiti [25]. On the other hand, when the site of the edge dislocation is at an interior point either in the half-plane or a general region Ω the complex potential (3.5.1), augmented by suitably chosen additional holomorphic complex potentials, does generate the solution to the problem of an edge dislocation.

As regards a wedge disclination, it follows from Section 2.3 that the constants A_k must always be present and consequently the complex potential $\varphi(z)$ must contain a part of the form (3.7.28). It is unlikely that appropriate complex potentials can be produced from the difference of two solutions to a problem for which A_k is absence from the corresponding complex potential and so the present procedure probably is ineffective. This remark may be illustrated by reference to the discussion of multi-valued displacements in a circular ring of inner and outer radii a and $b(< a)$ under no external loads. A solution presented by Muskhelishvili [2, p. 236] may be used for the inverse determination of the solution to a problem in the circular ring under prescribed boundary tractions but with multi-valued displacements. Accordingly, select the *difference* complex potentials to be given by

$$\varphi(z) = Az\log z + Az(B-1), \tag{3.8.1}$$

$$\psi(z) = 2ACz^{-1}, \tag{3.8.2}$$

where

$$A = \frac{\epsilon\mu}{4\pi(1-\nu^{(1)})}, \tag{3.8.3}$$

$$B = \frac{a^2(1-2\log a) - b^2(1-2\log b)}{2(a^2-b^2)}, \tag{3.8.4}$$

$$C = \frac{a^2b^2\log\frac{a}{b}}{a^2-b^2}. \tag{3.8.5}$$

As stated in [2, p. 236], the complex potentials (3.8.1) and (3.8.2) produce zero tractions on the inner and outer boundaries. Next, as in Section 3.4, we decompose the difference displacement into the constituent parts

$$D = D^\dagger + D^* \tag{3.8.6}$$

and let

$$\begin{aligned} 2\mu D^\dagger &= (3-4\nu^{(1)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ &= Az[2(1-2\nu^{(1)})\log r + 4(1-\nu^{(1)})\mathrm{i}\theta] + 2zAB(1-2\nu^{(1)}) \end{aligned} \tag{3.8.7}$$

$$-zA(3-4\nu^{(1)}) - 2ACzr^{-2}, \tag{3.8.8}$$

where $r^2 = z\bar{z}$. Consequently, the displacement $D^\dagger$ contains the dislocation

$$[D^\dagger]_C = \mathrm{i}\epsilon(x_1 + \mathrm{i}x_2). \tag{3.8.9}$$

where the simple curve C encloses the origin. The total difference displacement D is supposed single valued and so D^* must be chosen to possess a discontinuity equal and opposite to (3.8.7). Set

$$D^* = -\frac{\epsilon}{2\pi}z\log z + C_1 z, \tag{3.8.10}$$

where C_1 is a constant, and note by (3.6.1) that the associated thermoelastic problem possesses the temperature given by

$$T + \mathrm{i}S = 2[\frac{(\lambda^{(1)}+\mu)}{(3\lambda^{(1)}+2\mu)\kappa}][-\frac{\epsilon}{2\pi}(\log z + 1) + C_1]. \tag{3.8.11}$$

The coefficients ϵ and C_1 in (3.8.9) may be chosen to ensure that the temperature satisfies prescribed values on the inner and outer boundaries.

Also by (3.6.1) the complex potential $\varphi^{(2)}(z)$ in the traction boundary problem with Poisson ratio $\nu^{(2)}$ is given by

$$\varphi^{(2)}(z) = -[\frac{\mu}{2(\nu^{(1)} - \nu^{(2)})}][-\frac{\epsilon}{2\pi} z \log z + C_1 z], \tag{3.8.12}$$

and the complex potential $\varphi^{(1)}(z)$ for Poisson's ratio $\nu^{(1)}$ follows immediately. Observe that a term $z \log z$ necessarily is present in both complex potentials $\varphi^{(1)}$ and $\varphi^{(2)}$, so that, irrespective of the complex potentials $\psi^{(1)}$ and $\psi^{(2)}$, it may be concluded from (2.3.3) and (2.3.5) that both displacements $D^{(1)}$ and $D^{(2)}$ are multi-valued. The complex potential $\psi^{(2)}$ may be calculated from the prescribed boundary traction and enables the complex potential $\psi^{(1)}$ to be determined.

It is apparent that there is no advantage in considering the difference solution when treating wedge disclinations. Nevertheless, as noted by Muskhelishvili [2, p. 239], the solution with a wedge disclination does provide that for the steady state thermal problem with prescribed boundary temperatures and zero boundary tractions. The converse also holds.

4. Elastic inclusions

4.1. Introduction

This section seeks to demonstrate how the difference displacement and stress defined in Section 3.2 may be employed to solve the problem of elastic inclusions bonded to a matrix of different Poisson ratio but the same shear modulus. For this purpose, let the region Ω be singly connected and either bounded or unbounded, and let it contain simply connected internal regions that are separated from each other and do not intersect the external boundary $\partial\Omega$ of Ω. The interfacial contour between each internal region and its complement in Ω is a simple closed non-intersecting curve. The internal regions, conveniently termed inclusions, are each occupied by a different elastic material whose Poisson ratios differ from each other and from that of the elastic material in the multiply connected region (the matrix) which is the complement in Ω of the inclusions. The shear modulus, however, is the same in both inclusions and matrix. The region Ω is in equilibrium under zero body force and standard boundary conditions applied to the external boundary $\partial\Omega$. When the region is unbounded appropriate conditions are specified at infinity. Throughout the deformation, the inclusions remain bonded to the matrix in the sense that across each interfacial contour there is continuity of displacement and traction. Furthermore, the solution is supposed regular such that the complex potentials and derivatives are continuous onto each interfacial contour from both the matrix and respective inclusion, and onto the external boundary from the matrix. The displacement and stress are assumed single valued in both the inclusion and matrix.

4.2. The single inclusion

For simplicity of presentation, attention is restricted to only one inclusion which is denoted by Ω_1 and its Poisson ratio by $\nu^{(1)}$. The matrix is denoted by Ω_2 and its Poisson ratio is given by $\nu^{(2)}$. The shear modulus, as before, is μ, and C is the interfacial contour separating the inclusion and matrix. A similar procedure to the following may be applied to the general problem. Other generalisations are indicated later.

In the absence of the inclusion but with the boundary conditions unaltered, the problem reduces to the standard boundary-value problem for an elastic material of Poisson ratio $\nu^{(2)}$ and shear modulus μ occupying Ω. The corresponding complex potentials, denoted by $\varphi^{(2)}$ and $\psi^{(2)}$ to adhere to previously introduced notation, are holomorphic in the bounded region Ω. When Ω is the whole plane these functions, to comply with behaviour at infinity, assume the forms

$$\varphi^{(2)}(z) = \Gamma z, \qquad z \in \Omega, \tag{4.2.1}$$

$$\psi^{(2)}(z) = \Gamma^* z, \qquad z \in \Omega, \tag{4.2.2}$$

in which Γ and Γ^* are complex constants determined by prescribed conditions at infinity [2, p. 148]. As noted in Section 2.2, the complex potentials $\varphi^{(2)}$ and $\psi^{(2)}$, and consequently the stress components $\sigma^{(2)}_{\alpha\beta}$, are independent of the elastic moduli in the traction boundary-value problem.

When the inclusion is present, the complex potentials for both the inclusion and matrix are denoted by $\varphi^{(1)}$ and $\psi^{(1)}$. The assumed single valuedness of the displacement and stress everywhere in Ω_1 and Ω_2 implies that $\varphi^{(1)}$ and $\psi^{(1)}$ are holomorphic in Ω_1 and Ω_2 and continuous onto C from Ω_1 and Ω_2. Continuity of $\varphi^{(1)}$ and $\psi^{(1)}$ across C, however, cannot be assumed, and consequently $\varphi^{(1)}$ and $\psi^{(1)}$ are sectionally holomorphic in Ω with curve of discontinuity C. Independence from the elastic moduli no longer holds even in the traction boundary-value problem.

The difference complex potentials, which according to the previous definition are given by

$$\varphi(z) = \varphi^{(1)}(z) - \varphi^{(2)}(z), \qquad \psi(z) = \psi^{(1)}(z) - \psi^{(2)}(z), \tag{4.2.3}$$

inherit from $\varphi^{(1)}$ and $\psi^{(1)}$ the property of being sectionally holomorphic in Ω with curve of discontinuity C. The determination of $\varphi(z)$ and $\psi(z)$ is sought in terms of the complex potentials $\varphi^{(2)}(z)$ and $\psi^{(2)}(z)$, but before undertaking this task it is convenient to record expressions for the corresponding difference displacement and stress components for the inclusion and matrix. Let $D^{(1)}$ and $D^{(2)}$ be the displacement in Ω in the presence and absence of the inclusion and let $\sigma^{(1)}_{\alpha\beta}$ and $\sigma^{(2)}_{\alpha\beta}$ be the corresponding stress components. Introduce the definitions

$$D = D^{(1)} - D^{(2)}, \tag{4.2.4}$$

and

$$\sigma_{\alpha\beta} = \sigma^{(1)}_{\alpha\beta} - \sigma^{(2)}_{\alpha\beta}, \tag{4.2.5}$$

which in conjunction with (2.1.13)–(2.1.15) lead to the following expressions:

$$2\mu D = (3 - 4\nu^{(1)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - 4(\nu^{(1)} - \nu^{(2)})\varphi^{(2)}(z), \qquad z \in \Omega_1, \tag{4.2.6}$$

$$= (3 - 4\nu^{(2)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \qquad z \in \Omega_2, \tag{4.2.7}$$

and

$$\sigma_{\alpha\alpha} = 2(\varphi'(z) + \overline{\varphi'(z)}), \qquad z \in \Omega_1 \cup \Omega_2 \tag{4.2.8}$$

$$\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2(\bar{z}\varphi''(z) + \psi'(z)), \qquad z \in \Omega_1 \cup \Omega_2. \tag{4.2.9}$$

Note also that by (2.1.5) the difference dilatation $e_{\alpha\alpha}$ and rotation $\omega(=\omega^{(1)}-\omega^{(2)})$ satisfy

$$\mu e_{\alpha\alpha}+2\mathrm{i}\mu\omega=(3-4\nu^{(1)})\varphi'(z)-\overline{\varphi'(z)}-4(\nu^{1)}-\nu^{(2)})\varphi^{(2)'},\quad z\in\Omega_1, \tag{4.2.10}$$

$$=(3-4\nu^{(2)})\varphi'(z)-\overline{\varphi'(z)},\qquad z\in\Omega_2. \tag{4.2.11}$$

The determination of the complex potentials $\varphi(z)$ and $\psi(z)$ requires examination of their limiting behaviour as the contour C is approached from Ω_1 and Ω_2. In the absence of the inclusion, the displacement and traction are continuous across every simple curve in Ω, and when the inclusion is present, by hypothesis, the same quantities are continuous also across the contour C. Consequently, the difference displacement and traction are likewise continuous across C, which from (4.2.6) for the displacement implies:

$$(3-4\nu^{(1)})\varphi_+(s)-(3-4\nu^{(2)})\varphi_-(s)=4(\nu^{(1)}-\nu^{(2)})\varphi^{(2)}(s)+s[\overline{\varphi'_+(s)}-\overline{\varphi'_-(s)}]+\overline{\psi_+(s)}-\overline{\psi_-(s)}], \tag{4.2.12}$$

where a plus or negative subscript refers to quantities that approach their limiting behaviour on C from Ω_1 and Ω_2, respectively, s is a (complex) point on C, and C is described *anti-clockwise*. The difference traction across curves immediately adjacent to, but on either side of, C are obtained from (2.1.20) and respectively become:

$$F_1+\mathrm{i}F_2=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}[4(1-\nu^{(1)})\varphi_+(s)-4(\nu^{(1)}-\nu^{(2)})\varphi_+^{(2)}(s)-2\mu D_+],\quad z\in\Omega_1, \tag{4.2.13}$$

$$=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}[4(1-\nu^{(2)})\varphi_-(s)-2\mu D_-],\qquad z\in\Omega_2. \tag{4.2.14}$$

Continuity of traction across the contour C consequently yields:

$$(1-\nu^{(1)})\varphi_+(s)-(1-\nu^{(2)})\varphi_-(s)=(\nu^{(1)}-\nu^{(2)})\varphi^{(2)}(s),\qquad s\in C, \tag{4.2.15}$$

on recalling the continuity of D. On the external boundary $\partial\Omega$ either the difference displacement or traction vanish according to the prescription of the original boundary conditions. A similar remark applies to behaviour at infinity when the region Ω is unbounded.

The integration of sectionally holomorphic functions satisfying (4.2.15) leads to the following expression for the complex potential $\varphi(z)$ [2, p. 273, p. 444]:

$$\varphi(z)=\frac{(\nu^{(1)}-\nu^{(2)})}{(1-\nu^{(1)})}\varphi^{(2)}(z)+\frac{\varphi^o(z)}{(1-\nu^{(1)})},\qquad z\in\Omega_1, \tag{4.2.16}$$

$$=\frac{\varphi^o(z)}{(1-\nu^{(2)})},\qquad z\in\Omega_2, \tag{4.2.17}$$

where $\varphi^o(z)$ is a complex function holomorphic in Ω.

Substitution of (4.2.16) and (4.2.17) in (4.2.12) followed by a routine manipulation yields:

$$\psi_+(s)-\psi_-(s)=-\overline{h(s)},\qquad s\in C, \tag{4.2.18}$$

where

$$h(s)=\frac{(\nu^{(1)}-\nu^{(2)})}{(1-\nu^{(1)})}\left[\varphi^{(2)}(s)+s\overline{\varphi^{(2)'}(s)}+\frac{\varphi^o(s)+s\overline{\varphi^{o\prime}(s)})}{(1-\nu^{(2)}}\right], \tag{4.2.19}$$

which from the theory of sectionally holomorphic functions and the assumption $\psi(0) = 0$ leads after integration to:

$$\psi(z) = \psi^o(z), \qquad z \in \Omega_1, \tag{4.2.20}$$

$$= -\frac{1}{2i\pi}\oint_C \frac{\overline{h(s)}}{(s-z)}ds + \psi^o(z), \qquad z \in \Omega_2, \tag{4.2.21}$$

where $\psi^o(z)$ is a complex function holomorphic in Ω. The holomorphic functions $\varphi^o(z)$ and $\psi^o(z)$ are determined from boundary conditions on $\partial\Omega$ which, as just stated, vanish identically. For example, in the traction boundary-value problem for the bounded region Ω, the functions satisfy the condition:

$$\varphi^o(s) + s\overline{\varphi^{o'}(s)} + \overline{\psi^o(s)} = \frac{1}{2i\pi}\oint_{\partial\Omega} \frac{h(t)}{(\overline{s} - \overline{t})}dt, \tag{4.2.22}$$

which may be solved by standard techniques, first for the combination $\varphi^o(z) + z\overline{\varphi^{o'}(z)}$, and then for $\varphi^o(z)$ and $\psi^o(z)$. Details are omitted. Of course, when Ω occupies the whole space, and under the previously stated conditions, the holomorphic functions $\varphi^o(z)$ and $\psi^o(z)$ vanish identically by Liouville's theorem.

Accordingly, assume that the functions $\varphi^o(z)$ and $\psi^o(z)$ are known at all points of Ω. The complex potentials $\varphi^{(1)}$ and $\psi^{(1)}$ are obtained straightforwardly from (4.2.3), (4.2.16), (4.2.17), (4.2.20) and (4.2.21) and are given by:

$$\varphi^{(1)}(z) = \frac{(1-\nu^{(2)})}{(1-\nu^{(1)})}\varphi^{(2)}(z) + \frac{\varphi^o(z)}{(1-\nu^{(1)})}, \qquad z \in \Omega_1, \tag{4.2.23}$$

$$= \varphi^{(2)}(z) + \frac{\varphi^o(z)}{(1-\nu^{(2)})}, \qquad z \in \Omega_2, \tag{4.2.24}$$

and

$$\psi^{(1)}(z) = \psi^{(2)}(z) + \psi^o(z), \qquad z \in \Omega_1, \tag{4.2.25}$$

$$= -\frac{1}{2\pi i}\oint_C \frac{\overline{h(s)}}{(s-z)}ds + \psi^{(2)}(z) + \psi^o(z), \qquad z \in \Omega_2. \tag{4.2.26}$$

The displacement and stress components when the inclusion is present follow by application of (2.1.13)–(2.1.15) to the complex potentials $\varphi^{(1)}$ and $\psi^{(1)}(z)$. Consequently, under the stated conditions, the inclusion problem is completely solved in terms solely of the complex potentials $\varphi^{(2)}(z)$, $\varphi^o(z)$ and $\psi^o(z)$. Note that the method of solution is independent of the size and shape of the inclusion, the geometry of Ω, and of the boundary conditions on the external boundary (or at infinity). The method may be extended to several bonded inclusions whose Poisson ratios differ from each other and from that of the matrix.

Expressions are next sought for the dilatation and rotation at points in Ω_1 and Ω_2 , and it is convenient to consider only unbounded regions Ω for which, as noted earlier, the complex potentials $\varphi^o(z)$ and $\psi^o(z)$ are identically zero. Substitution from (4.2.16), (4.2.17) in (4.2.10) and (4.2.11) yields

$$\mu e_{\alpha\alpha} + 2i\mu\omega = -\frac{(\nu^{(1)} - \nu^{(2)})}{(1-\nu^{(1)})}(\varphi^{(2)'}(z) + \overline{\varphi^{(2)'}(z)}), \qquad z \in \Omega_1, \tag{4.2.27}$$

$$= 0, \qquad z \in \Omega_2, \tag{4.2.28}$$

from which it follows that the inclusion does not alter the rotation anywhere in Ω nor the dilatation except in the region of the inclusion itself. To determine the dilatation inside the inclusion, note first from (2.1.11) and (2.1.14) that

$$2(\lambda^{(2)} + \mu)e_{\alpha\alpha}^{(2)} = \sigma_{\alpha\alpha}^{(2)} = 2(\varphi^{(2)^i}(z) + \overline{\varphi^{(2)^{\prime}}(z)}), \tag{4.2.29}$$

which, on combining with (4.2.27) and (4.2.28), leads to

$$e_{\alpha\alpha} = -\frac{(\nu^{(1)} - \nu^{(2)})}{(1 - \nu^{(1)})(1 - 2\nu^{(2)})} e_{\alpha\alpha}^{(2)}, \qquad (x_1, x_2) \in \Omega_1, \tag{4.2.30}$$

so that the final dilatation at points inside the inclusion is expressed by

$$e_{\alpha\alpha}^{(1)} = e_{\alpha\alpha} + e_{\alpha\alpha}^{(2)} = \frac{(\lambda^{(2)} + 2\mu)}{(\lambda^{(1)} + 2\mu)} e_{\alpha\alpha}^{(2)}, \qquad (x_1, x_2) \in \Omega_1. \tag{4.2.31}$$

It follows that the expansion of the region occupied by the inclusion is given by

$$\int_{\Omega_1} e_{\alpha\alpha}^{(1)} \mathrm{d}x_1 \mathrm{d}x_2 = \frac{(\lambda^{(2)} + 2\mu)}{(\lambda^{(1)} + 2\mu)} \int_{\Omega_1} e_{\alpha\alpha}^{(2)} \mathrm{d}x_1 \mathrm{d}x_2. \tag{4.2.32}$$

4.3. Discussion

The three-dimensional formula analogous to (4.2.32) has been derived in [26, 27] by an argument similar to that described here. The discussion in [27] permits both elastic moduli to be simultaneously varied and besides the derivation of the exact solution also presents expressions for the mean values of the strains. The technique is readily adapted to the two-dimensional problem and is not repeated here. Other treatments of the three-dimensional inclusion problem permitting variation in both moduli are, for example, due to Eshelby [6], Kupradze [28] and Hill [29]. The last author is concerned with establishing bounds and relations for the moduli important in the theory of elastic composites, and which of course apply equally to problems in plane elasticity.

Eshelby's approach is to interpret the problem as the insertion of an inclusion into a misfitting cavity contained in the matrix. It is instructive to sketch the argument in the present context as it leads also to a possible relation with dislocations. A homogeneous region Ω_1 of Poisson ratio $\nu^{(1)}$ and shear modulus μ is to be inserted into a cavity in the unstressed matrix Ω_2 of Poisson ratio $\nu^{(2)}$ and the same shear modulus. The amount of displacement that must be applied to each surface point of Ω_1 to ensure that it perfectly fits into the cavity is D^* given by (3.4.2). Once inserted, the region Ω_1 is bonded to the matrix and becomes the inclusion, while the mechanism for applying the surface displacement D^* is relaxed creating the additional displacement $D^{\dagger} = D - D^*$ and the associated stress $\sigma_{\alpha\beta}$. Because by construction the final displacement D is continuous across the interface C between the inclusion and matrix, there is a discontinuity in the displacement $D^{\dagger}$ of amount $-D^*$ which is identified by Dundurs [14] as a Somigliana dislocation (see also Eshelby [30] and Maiti [25]). The two-dimensional problem posed in this manner was first solved by Sherman [31] (see also [2, p. 442]) by a different method. An alternative treatment using the so-called Betti-Somigliana integral is provided by Maiti and Makan [32], while a power series expansion solution is presented by

Buchwald and Davies [33]. A recent direct treatment of a slightly more general problem is by Shen *et al.* [34]; see also Ru [35].

The inclusion problem may also be interpreted to yield a relationship with the thermoelastic problem of a region Ω_1, heated to a steady temperature T given by (3.5.2), and bonded to the matrix Ω_2 maintained at zero temperature and subject to zero boundary conditions on the outer boundary $\partial\Omega$. The corresponding thermal displacement and stresses are then the difference displacement and stresses (4.2.6), (4.2.7), (4.2.8) and (4.2.9). The reverse is also possible. The temperature distribution in the inclusion determines the real part of the function $\varphi^{(2)'}(z)$ from which the imaginary part may be calculated in the standard way. Thereafter, analytic continuation enables the function $\varphi^{(2)'}(z)$ to be determined in Ω_2. The holomorphic function $\psi^{(2)}(z)$ then may be found upon prescribing the boundary conditions on the external boundary $\partial\Omega$ and consequently the displacement $D^{(2)}$ and the stress components $\sigma^{(2)}_{\alpha\beta}$ are known. The final displacement $D^{(1)}$ and stress components $\sigma^{(1)}_{\alpha\beta}$ are found from (4.2.4) and (4.2.5).

4.4. Example

We conclude this section with the simple example of a circular inclusion of radius a contained in an infinite matrix subject to uniform pressure p at infinity. (*cp.*, [2, p. 220]). The complex potentials in the absence of the inclusion are given by

$$\varphi^{(2)}(z) = \frac{1}{4}pz, \qquad \psi^{(2)}(z) = -\frac{1}{2}pz, \quad z \in \Omega. \tag{4.4.1}$$

By (4.2.23)–(4.2.26), the final complex potentials become

$$\varphi^{(1)}(z) = \frac{pz(1-\nu^{(2)})}{4(1-\nu^{(1)})}, \qquad \psi^{(1)}(z) = -\frac{1}{2}pz, \qquad z \in \Omega_1, \tag{4.4.2}$$

$$\varphi^{(1)}(z) = \frac{1}{4}pz, \qquad \psi^{(1)}(z) = -\frac{p[z + a^2(\nu^{(1)} - \nu^{(2)})]}{z(1-\nu^{(1)})}, \qquad z \in \Omega_2, \tag{4.4.3}$$

and the displacement and stress may be calculated as before.

5. Conclusion

It is beyond the present scope to provide a comprehensive treatment of all problems in the category under consideration. The intention instead has been to indicate a possible unified strategy for the plane problem and, incidentally, to use the results to complement and contrast those for corresponding three-dimensional problems. Such comparison also suggests further studies. In the plane theory, for example, the relationships developed might usefully be exploited as an alternative method of investigating the force on a defect and subsequent connexion with the energy momentum tensor. The solution to the problem of a continuous distribution of dislocations might also be amenable to the present approach. It appears unlikely, however, that problems with a space-dependent Poisson ratio can be accommodated, although the additional variation of a uniform shear modulus can be included in the analysis. In the previous notation the difference displacement (3.2.3) becomes

$$2\mu^{(1)}D = (3-4\nu^{(1)})\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \left[4(\nu^{(1)}-\nu^{(2)}) + \frac{(\mu^{(1)}-\mu^{(2)})(3-4\nu^{(2)})}{\mu^{(2)}}\right]\varphi^{(2)}(z) + \left[\frac{(\mu^{(1)}-\mu^{(2)})}{\mu^{(2)}}\right][z\overline{\varphi^{(2)'}(z)} + \overline{\psi^{(2)}(z)}],$$

with appropriate modification for the difference stresses. A similar analysis to that described here is applicable and will be presented elsewhere.

A further relationship may be developed with the notion of internal stress. Details are omitted because the connexion between thermoelasticity, dislocations and internal stress is adequately treated in standard texts (*e.g.*, [4, p. 425]) and can be incorporated easily into the present analysis. Note also that experimental tests for dislocations are described in the books by Nabarro [12] and Mura [36].

Finally, it is worth remarking that the scope of possible applications can be enlarged by combining the present procedure with, for example, the analogy between elastic plane strain and transverse flexure of a thin plate (Mindlin [37]), or with the thermoelastic similarity laws discussed by Green, Radok and Rivlin [38].

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